



TOPIC 1: MEASUREMENT

Lecturer: Mr. Lim Teck Seng

Content

1. Physical Quantities & SI Units
2. Errors and uncertainties
3. Scalars and vectors

Learning Outcomes

Candidates should be able to:

- (a) recall the following base quantities and their units: mass (kg), length (m), time (s), current (A), temperature (K), amount of substance (mol).
- (b) express derived units as products or quotients of the base units and use the named units listed in 'Summary of Key Quantities, Symbols and Units' as appropriate.
- (c) show an understanding of and use the conventions for labelling graph axes and table columns as set out in the ASE publication (*SI Units, Signs, Symbols and Abbreviations*), except where these have been superseded by (*Signs, Symbols and Systematics (The ASE Companion to 5-16 Science, 1995)*).
- (d) use the following prefixes and their symbols to indicate decimal sub-multiples or multiples of both base and derived units: pico (p), nano (n), micro (μ), milli (m), centi (c), deci (d), kilo (k), mega (M), giga (G), tera (T).
- (e) make reasonable estimates of physical quantities included within the syllabus.
- (f) show an understanding of the distinction between systematic errors (including zero errors) and random errors.
- (g) show an understanding of the distinction between precision and accuracy.
- (h) assess the uncertainty in a derived quantity by simple addition of actual, fractional or percentage uncertainties (a rigorous statistical treatment is not required).
- (i) distinguish between scalar and vector quantities, and give examples of each.
- (j) add and subtract coplanar vectors.
- (k) represent a vector as two perpendicular components.

Reference

Text: Longman A-level Course in Physics

Website: <http://www.rit.edu/~uphysics/uncertainties/Uncertaintiespart2.html>

http://www.batesville.k12.in.us/physics/APPhyNet/Measurement/Measurement_Intro.html

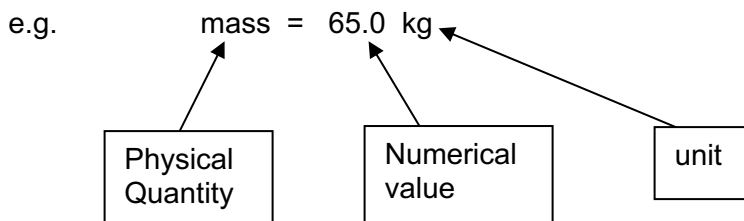
1 Physical Quantities and SI Units

What is a physical quantity?

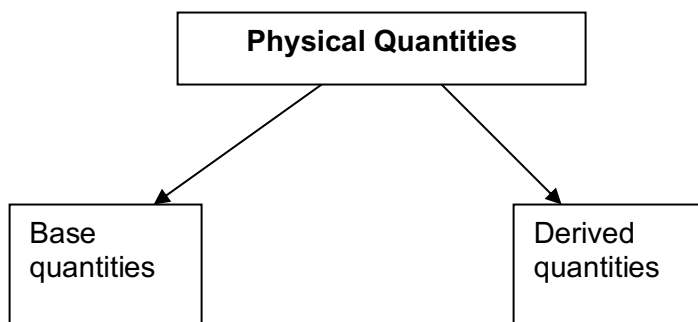
Any scientifically measurable quantities are called physical quantities.
e.g. temperature, force, electric current, pressure, mass, energy, etc.

What are features of physical quantities?

All physical quantities consist of a numerical value and a unit.



AO(a) recall the following base quantities and their units: mass (kg), length (m), time (s), current (A), temperature (K), amount of substance (mol).



What is a base quantity?

The seven physical quantities of the SI system by which all other physical quantities are defined.

They are: mass, length, time, temperature, amount of substance, electric current and luminous intensity.

Summary of the base physical quantities, units and symbols:

Base Quantity	Base unit	Symbol	Definition
Mass (m)	Kilogramme	kg	The unit of mass is equal to the mass of the international prototype kilogram (a platinum-iridium cylinder) kept at the Bureau International des Poids et Mesures (BIPM), Sevres, Paris.
Length (l)	Metre	m	The unit of length is equal to the length of the path travelled by light in a vacuum during the time interval of $1/299\,792\,458$ of a second.
Time (t)	Second	s	The unit of time is the duration of exactly 9 192 631 770 periods of the radiation corresponding to the transition between two hyperfine levels of the ground state of the caesium-133 atom at a temperature of 0 K.
electric current (I)	Ampere	A	The unit of electrical current is the constant current which, if maintained in two straight parallel conductors, of infinite length and negligible cross-section, placed 1 m apart in a vacuum, would produce a force between these conductors equal to 2×10^{-7} N per metre of length.
thermodynamic temperature (T)	Kelvin	K	The unit of thermodynamic temperature (or absolute temperature) is the fraction $1/273.16$ of the difference between the absolute zero of temperature and thermodynamic temperature at triple point of water. *
amount of substance (n)	Mole	mol	A mole is the amount of substance that contains the number of molecules (or atoms if the substance is monoatomic) equal to Avogadro Constant. Avogadro Constant is the number of atoms in 0.012 kg of pure carbon-12; this number (N_A) is approximately equal to $6.02214199 \times 10^{23}$. *
luminous intensity (not in syllabus)	Candela	cd	The unit of luminous intensity is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} Hz and that has a radiant intensity in that direction of $1/683$ watt per steradian.

Note: the units are expressed in small letters when written in full.

*Need to memorise

AO (b) express derived units as products or quotients of the base units and use the named units listed in 'Summary of Key Quantities, Symbols and Units'¹ as appropriate.

What are derived units?

Derived units are units that are derived from base units and can be expressed in terms of products and quotients of base units.

¹ Refer to Appendix A.

Likewise, a derived quantities are physical quantities that are derived from base quantities and can be expressed in terms of products and quotients of base quantities.

Examples:

Derived quantity	Obtained from	Derived unit
Density	$\frac{\text{mass}}{\text{volume}}$	$\frac{\text{kg}}{\text{m}^3} = \text{kgm}^{-3}$
Velocity	$\frac{\Delta \text{displacement}}{\Delta \text{time}}$	ms^{-1}
Kinetic energy	$\frac{1}{2} \times \text{mass} \times \text{velocity}^2$	$\text{kg}(\text{ms}^{-1})^2 = \text{kgm}^2\text{s}^{-2}$
Frequency	Number of cycles per unit time	s^{-1}
Force	mass x acceleration	kgms^{-2}
Pressure	$\frac{\text{force}}{\text{area}}$	$\text{kgm}^{-1}\text{s}^{-2}$
Charge	Current x time	As
Potential difference	$\frac{\text{work}}{\text{charge}}$	$\text{Kgm}^2\text{A}^{-1}\text{s}^{-3}$

Unitless quantities

- all numbers, e.g. 2, $\frac{1}{2}$, π , e
- trigonometrical functions, e.g. sine, cosine, tangent
- all logarithmic functions, e.g. \log_x , \ln
- powers, e.g. $10^{\frac{x}{y}}$, the ratio $\frac{x}{y}$ must be unitless. If x has a unit, then the unit of y must have the same unit as that of x.
- Unitless physical constants: e.g. refractive index of glass, relative density of a liquid.

Uses of Base Units

- To find units of unknown quantities in an equation.
- To check homogeneity of a physical equation.

If units of each term on LHS = units of each term on RHS, the equation is said to be *homogeneous* or *dimensionally consistent*.

Note: Checking homogeneity does not guarantee that the equation is physically correct!

Use base units to check the homogeneity of equation.

An equation is homogeneous if each of the terms in the equation can be expressed using the same units.

A physically correct equation must always be homogeneous. However, a homogeneous equation need not be physically correct. Why?.

There are two basic reasons:

- (1) The value of the coefficient may be incorrect.

e.g. $E = 3mv^2$ where $E =$ kinetic energy

The coefficient 3 is incorrect! The value should be $\frac{1}{2}$ instead.

- (2) Missing or extra terms that may have the same unit.

e.g. $E = \frac{1}{2}mv^2 + mgh$ where $E =$ kinetic energy

There is an extra term mgh , which happens to have the same unit as kinetic energy. This is an extra term.

Example: Bernoulli's equation, which applies to fluid flow states that

$$p + h\rho g + \frac{1}{2}\rho v^2 = k$$

Where p is pressure, h is height, ρ is density, g is acceleration of free fall, v is velocity and k is constant. Show the equation is dimensionally consistent and state the SI unit for k .

Unit of $p = \text{Nm}^{-2} = \text{kgms}^{-2}\text{m}^{-2} = \text{kg m}^{-1}\text{s}^{-2}$

Unit of $h\rho g = \text{m}(\text{kgm}^{-3})(\text{ms}^{-2}) = \text{kg m}^{-1}\text{s}^{-2}$

Unit of $\frac{1}{2}\rho v^2 = (\text{kgm}^{-3})(\text{m}^2\text{s}^{-2}) = \text{kg m}^{-1}\text{s}^{-2}$

Since all terms have the same unit, the equation is dimensionally consistent.
SI unit for $k = \text{kg m}^{-1}\text{s}^{-2}$

NOTE: Homogeneous equation is not necessary. physically correct.

AO(c) show an understanding and use the conventions for labelling graph axes and table columns.

Conventions for Labelling Table Columns

- All readings or measurements should be tabulated in vertical columns;

For example

<i>V / V</i>	<i>I / mA</i>	<i>t / s</i>	<i>a / ms⁻²</i>	<i>m / kg</i>	<i>lg (m / kg)</i>
3.0	0.30	2.0	5.0	2.500	0.3979

- For the column headings, use the standard notation of “**quantity / unit**”; it means “**quantity divided by unit**”.

For example

Writing $t = 2.0 \text{ s}$ as $t / \text{s} = 2.0$,

the expression becomes a pure number. The column with t / s thus consists of just pure numbers with no units.

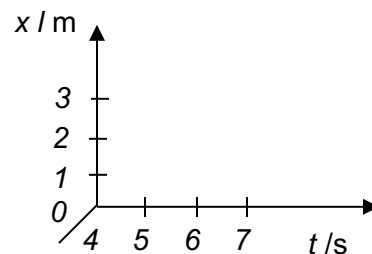
- The unit should be written in the index form, e.g. use ms^{-2} and not m / s^2 .
- For columns that involve logarithms, either \lg or \ln , the unit of the variable must be stated.

For example, $\lg(m / \text{kg})$, $\ln(T / \text{s})$.

It should be noted that after taking \lg or \ln , the resulting values have no units.

Conventions for Labelling Graph Axes

- when plotting graphs, both axes must be labelled with the physical quantities and their associated units (e.g. x / m and t / s). Therefore, there is no need to write unit to every number label on the axes.



AO(d) use the following prefixes and their symbols to indicate decimal sub-multiples or multiples of both base and derived units: pico (p), nano (n), micro (μ), milli (m), centi (c), deci (d), kilo (k), mega (M), giga (G), tera (T).

10^n	Prefix	Symbol	Name	Decimal equivalent
10^{-12}	pico	p	Trillionth	0.000,000,000,001
10^{-9}	nano	n	Billionth	0.000,000,001
10^{-6}	micro	μ	Millionth	0.000,001
10^{-3}	milli	m	Thousandth	0.001
10^{-2}	centi	c	Hundredth	0.01
10^{-1}	deci	d	Tenth	0.1
10^0	-	-	One	1
10^3	kilo	k	Thousand	1,000
10^6	mega	M	Million	1,000,000
10^9	giga	G	Billion	1,000,000,000
10^{12}	tera	T	Trillion	1,000,000,000,000

AO(e) make reasonable estimates of physical quantities within the syllabus.

The following are compilations of some estimates: (not exhaustive)

Mass	Approximately (kg)	Scientific notation	Order of magnitude
Tennis ball	0.057	5.7×10^{-3}	-3
An apple	0.350	3.50×10^{-1}	-1
Adult Asian man	75	7.5×10^1	1
Car	1000 - 1500	1.0×10^3	3
SBS bus	10,000	1.0×10^4	4
Earth	6.0×10^{24}	6.0×10^{24}	24
Length	Approximately (m)	Scientific notation	Order of magnitude
Diameter of hair	0.0002 (0.2 mm)	2×10^{-4}	-4
Finger	0.070 (7.0 cm)	7.0×10^{-2}	-2
Width of a car	1.7	1.7×10^0	0
Football field	100	1.0×10^2	2
Radius of Earth	6.4×10^6	6.4×10^6	6
Speed	Approximately (ms^{-1})	Scientific notation	Order of magnitude
Fastest runner	10	1.0×10^1	1
Sound in air	330	3.3×10^2	2
Light in vacuum (c)	3.00×10^8	3.00×10^8	8
α -particle	0.1 c	3.00×10^7	7
β -particle	0.9 c	2.70×10^8	8
Temperature	Approximately (K)	Scientific notation	Order of magnitude
Room (Singapore)	306 (33°C)	3.06×10^2	2
Candle flame	1000 – 1700		3
Blue Bunsen flame	3000	3.00×10^3	3
Tested in A level:	accelerating p.d. in cathode ray tube:	kV or 10^3 V	
	Speed limit of Singapore expressway:	80 kmh^{-1}	
	Power of domestic light	40 to 100 W	

2 Errors and Uncertainties

Uncertainty is the range of values on both sides of a measurement in which the actual value of the measurement is expected to lie. It is determined by the graduation of scale of the measuring instrument.

2.1 Experimental Errors

Error is the difference between the measured value and the 'true value'.

For example, the accepted value of acceleration due to gravity g at a certain location is 9.81ms^{-2} . If an experimental determination yields a result of 9.9ms^{-2} , the error is 0.1ms^{-2} .

Note:

- Blunders such as misreading a ruler or calculation mistakes should NOT be quoted as errors in this context!
- (Absolute) Errors are estimates and hence they are expressed only to one significant figure.
- Uncertainties in measured quantities arise from:
 - limitations of the observer;
 - instrumental limitations;
 - the method used (the experimental design).

2.2 Types of Experimental Errors

AO(f) show an understanding of the distinction between systematic errors (including zero errors) and random errors.

In assessing errors, whether human or instrumental, there are two types of errors:

(1) systematic errors and (2) random errors

(1) Systematic errors are consistent deviation of readings from the true value with a fixed pattern. They have the same magnitude and sign.

Systematic errors are more serious forms of errors since they cannot be corrected by taking repeated readings or by any other form of averaging.

However, they can be reduced or eliminated by careful design of an experiment, and good experimental techniques.

Examples of Systematic Errors:

Error Sources	Descriptions	Corrections
– Due to apparatus	<ul style="list-style-type: none"> • zero errors on the scales of instruments • poor calibration of instruments 	<ul style="list-style-type: none"> • correct all measured readings by negating the error accordingly. • calibrate the instrument properly before experiment

– Due to poor experimental technique	<ul style="list-style-type: none"> • consistent parallax error which affect all the readings in the same way, for instance, taking readings off a scale from a fixed angle. 	<ul style="list-style-type: none"> • adopt the correct way to take reading: ensure that the <i>line of sight</i> is <i>perpendicular</i> to the measuring scale.
– Due to external factors	<ul style="list-style-type: none"> • <i>background radiation</i> causes the count rate of your radioactive sample to be consistently higher than the true reading. 	Take the external factor(s) into account and adjust all readings appropriately. For instance, <ul style="list-style-type: none"> • Measure the count rate of background radiation and subtract it from the readings.

(2) **Random errors** are present when the measured readings are scattered about the mean value with no fixed pattern. They have equal probability of having different magnitudes and signs.

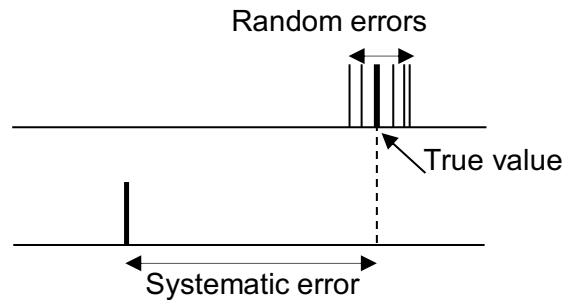
Random errors are of varying signs and magnitudes and cannot be eliminated, but they can usually be reduced by taking average of repeated readings or using other averaging methods.

Examples of Random Errors:

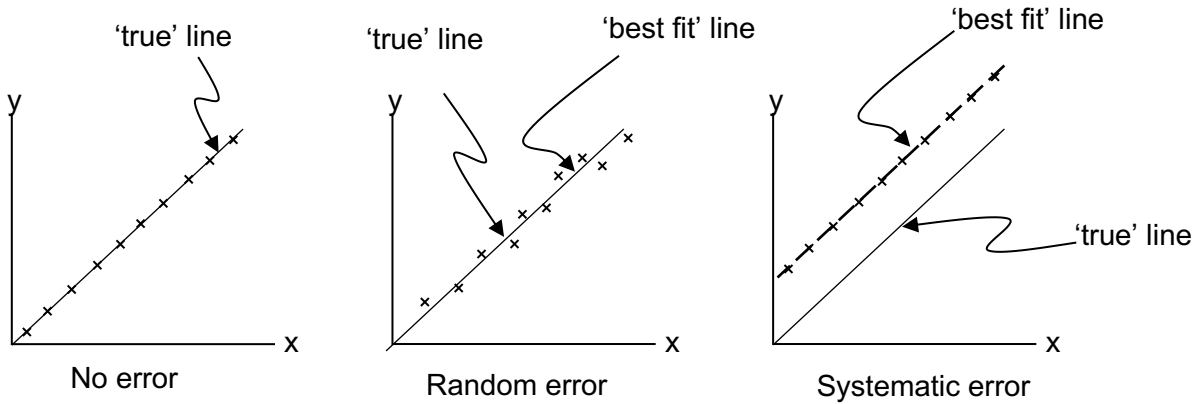
Error Sources	Descriptions	Methods to reduce error
– Due to inability of observer to repeat his action precisely	<ul style="list-style-type: none"> • inconsistent reaction time when using stop watch to measure the period of an oscillation. • random parallax error, when the line of sight of observer is sometimes on one side of the perpendicular to the scale and sometimes on the other side. 	<ul style="list-style-type: none"> • To reduce the random errors, take the time of 20 oscillations then find the mean period. • Can be avoided if observer's line of sight is perpendicular to the scale. (i.e. aligning his eyes with the pointer and scale).
– Due to environmental conditions like pressure, temperature	<ul style="list-style-type: none"> • fluctuations in the measurement (of length, time, current, etc.) 	<ul style="list-style-type: none"> • Random errors can be reduced by taking average of repeated readings.
– Due to the limited sensitivity of instruments.	<ul style="list-style-type: none"> • Precision of metre rule: 1mm vernier calliper: 0.1mm micrometer screw gauge: 0.01 mm 	<ul style="list-style-type: none"> • To measure the thickness of a coin with only a metre rule, the random errors can be reduced by determining the mean thickness of a stack of 10 identical coins. (Use a micrometer screw gauge if it is available).

Distinguish between systematic and random errors using Illustrations.

The figure shows a spread of readings caused by random errors; these are approximately centred about a mean value which coincides with the *true value*. If a systematic error is present, the mean value will be shifted.

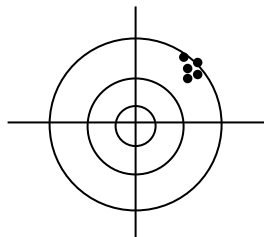


The effects of random errors and of systematic errors appear in graphs as illustrated below.

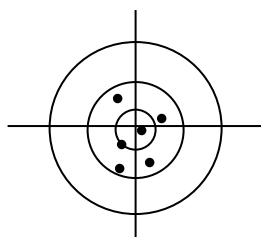


2.3 Precision and Accuracy

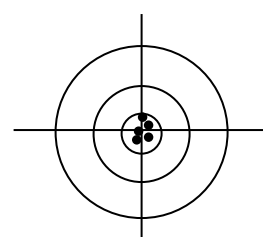
AO(g) show an understanding of the distinction between precision and accuracy.



Good precision,
Poor accuracy
(small random error, but large systematic error!)



Good accuracy,
Poor precision
(large random error, small systematic error)



Good precision and accuracy
(small random error, small systematic error)

The **precision** of a measurement is how close the experimental values are to each other. Precision is also a term used to describe the level of uncertainty in an instrument's scale.

Good precision means the readings are mostly very *close to their mean*, and is associated with small random errors.

Accuracy is the closeness of experimental value to the true value of the quantity being measured.

Good accuracy means the reading or the mean of a set of readings is very close to the true value, and is associated with small systematic errors.

Illustration:

The first few decimal places of the true value for the mathematical constant π are 3.142, and the accepted value for the speed of light in a vacuum is $2.99792458 \times 10^8 \text{ ms}^{-1}$.

Thus, 3.14 is an **accurate** value for π to three digits **precision**, and $3.0 \times 10^8 \text{ ms}^{-1}$ is an **accurate** value for the speed of light in a vacuum to two digits **precision**.

Worked Example 1

Student A carried out a series of experiments to obtain the value of the Earth's gravitational acceleration g . He was very careless and did not take into account the mass of the mass holder. This resulted in all recorded values of the mass being smaller than the actual mass. He obtained the following values of g :

Reading/ ms^{-2}	10.33	10.32	10.31	10.30	10.29
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Given that the actual value of $g = 9.81 \text{ ms}^{-2}$, comment on the accuracy and precision for this experiment.

Answer: mean = $(10.33 + 10.32 + 10.31 + 10.30 + 10.29) / 5 = 10.31 \text{ ms}^{-2}$

The readings are quite precise because there is only a small spread about the mean value.
The readings are not accurate because the mean value deviates from the true value.

Worked Example 2

Student B measured the gravitational constant g a number of times and got the following results

Reading/ ms^{-2}	12.20	9.81	7.42	9.99	9.63
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How do these results compare to those of Student A in worked example 1?

Student B's mean reading = $(12.20 + 9.81 + 7.42 + 9.99 + 9.63) / 5 = 9.81 \text{ ms}^{-2}$

Student A's mean reading = $10.31 \text{ ms}^{-2} > 9.81 \text{ ms}^{-2}$

Since the Student B's mean reading is equal to the true value, 9.81 ms^{-2} , Student B's readings are more accurate.

Range of Student B's readings = Largest reading - smallest reading = $12.20 - 7.42 = 4.78 \text{ ms}^{-2}$

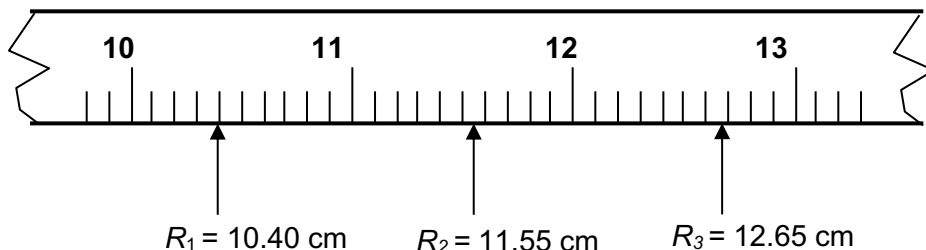
Range of Student A's readings = $10.33 - 10.29 = 0.04 \text{ ms}^{-2}$

Since Student A's readings have a smaller spread, which implies smaller random errors. His readings are more precise.

2.4 Absolute and Relative (Fractional and Percentage) Uncertainties

2.4.1 (Absolute/ Actual) Uncertainty in a Reading

- A reading is the single determination of a value at one point on a measuring scale.
- Generally a **reading** can be estimated to *half* of the smallest division on a measuring scale.



- The (actual or absolute) *uncertainty* in the reading of an instrument is thus taken as *half the smallest division*. In the above illustration, the *absolute uncertainty* is 0.05 cm.
- To indicate the magnitude of uncertainty, $R_1 = 10.40 \text{ cm}$ in the above illustration is written as $R_1 = (10.40 \pm 0.05) \text{ cm}$. This means that R_1 can take values in the range between 10.35 cm to 10.45 cm.
- The absolute uncertainty 0.05 cm is also known as the *maximum uncertainty* in the reading.
- The absolute uncertainty should always be rounded off to 1 significant figure only.
- In general, all readings can be recorded in the form $R \pm \Delta R$ in which ΔR is the absolute uncertainty. R should be rounded to the same number of decimal places as the uncertainty ΔR . For example $(10.0 \pm 0.1) \text{ cm}$ should *not* be written as $(10 \pm 0.1) \text{ cm}$

Relative Uncertainties

The uncertainty of a measured value can also be presented as a percent or a simple fraction.

(a) The fractional uncertainty of $R = \frac{\Delta R}{R}$

(b) The percentage uncertainty of $R = \frac{\Delta R}{R} \times 100\%$

For example, the measurement $(208 \pm 1) \text{ mm}$,

$$\text{Absolute uncertainty} = 1 \text{ mm (1 s.f.)}$$

$$\text{Fractional uncertainty} = \frac{1}{208} = 0.0048 \quad (2 \text{ s.f.})$$

$$\text{Percentage uncertainty} = 0.0048 \times 100\% = 0.48\% \quad (2 \text{ s.f.})$$

Note that the absolute uncertainty has units, whereas fractional and percentage uncertainties are ratios and therefore are dimensionless.

While the absolute uncertainty is an indication of the scale sensitivity or the accuracy of the measuring instrument used, the percentage error is useful to compare the absolute error and the quantity value to tell whether the error is negligible or significant.

Worked Example 1

A metre rule has a precision to 1 mm, vernier callipers to 0.1 mm and a micrometer screw gauge to 0.01 mm. If these instruments were to be used to measure the diameter of a wire which is 2.5 mm, what would be the respective percentage uncertainties?

For ruler, % uncertainty = $(1/2.5) \times 100\% = 40\%$

For vernier callipers, % uncertainty = $(0.1/2.5) \times 100\% = 4\%$

For micrometer % uncertainty = $(0.01/2.5) \times 100\% = 0.4\%$

Worked Example 2

A student makes measurements from which he calculates the speed of sound as 327.66 ms^{-1} . He estimates that his result contains a percentage error of $\pm 3\%$. Give his result reduced to the appropriate number of significant figures.

$$\frac{\Delta v}{v} \times 100\% = 3\%$$

$$\frac{\Delta v}{v} = 0.03$$

$$\Delta v = 0.03 \times 327.66 = 10 \text{ ms}^{-1} \quad (\text{to 1 s.f.})$$

$$v \pm \Delta v = (330 \pm 10) \text{ ms}^{-1} \quad (v \text{ expressed to same d.p as } \Delta v)$$

2.4.2 Propagation of Uncertainties

AO (h) assess the uncertainty in a derived quantity by simple addition of actual, fractional or percentage uncertainties (a rigorous statistical treatment is not required).

The result of an experiment is seldom obtained by a single measurement; very often it is obtained by measuring a few related quantities. The overall estimate of uncertainty is called the **consequential uncertainty**.

There are established statistical rules for calculation of consequential uncertainty from individual pieces of information. The A-level course only requires a simplified version of the statistical treatment.

The guiding principle in all cases is to consider the maximum uncertainty i.e. the worst case scenario.

Basic Rules of Consequential Uncertainties at a Glance

Suppose A and B are measured quantities, ΔA and ΔB are the corresponding (actual) uncertainties.

Given that n and m are numerical constants, e.g. $\frac{1}{2}$, 3 , π .

Addition

If $R_1 = A + B$, then $\Delta R_1 = \Delta A + \Delta B$

Subtraction

If $R_2 = A - B$, then $\Delta R_2 = \Delta A + \Delta B$

} Adding absolute uncertainties

Multiplied by a constant

If $R_3 = nA$, then $\Delta R_3 = |n|\Delta A$

$$\text{Note: } \frac{\Delta R_3}{R_3} = \frac{\Delta A}{A}$$

Product

If $R_4 = A \times B$

$$\frac{\Delta R_4}{R_4} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$$

Quotient

If $R_5 = \frac{A}{B}$

$$\frac{\Delta R_5}{R_5} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$$

} Adding relative uncertainties

Power

If $R_6 = A^n$, then $\frac{\Delta R_6}{R_6} = |n| \frac{\Delta A}{A}$

Summary

1. If $Z = mA + nB$, then $\Delta Z = |m|\Delta A + |n|\Delta B$ Note: m, n are numerical constants
 2. If $Z = kA^m \times B^n$, where k is a numerical constant, then $\frac{\Delta Z}{Z} = |m|\frac{\Delta A}{A} + |n|\frac{\Delta B}{B}$
 3. Other functions: e.g. $Z = \sin A$, $Z = \ln A$
- Use the general approach: $\Delta Z = \frac{1}{2}(Z_{\max} - Z_{\min})$

Propagation Rules of Consequential Uncertainties – Explained.

In the following discussion, we consider $A \pm \Delta A$, $B \pm \Delta B$ and $C \pm \Delta C$ as *measured quantities*; k, m, n are numerical constant, e.g. $\frac{1}{2}, 3, \pi$.

Uncertainties are treated as relatively small values so that the approximations can be justified.

1. Addition and Subtraction of Measurements

When adding or subtracting measurements, add their absolute uncertainties.

$$Z = A + B \quad \text{or} \quad Z = A - B$$

The maximum uncertainty, $\Delta Z = \Delta A + \Delta B$ for both cases.

Suppose we need to add the measurements $A: (20 \pm 2)$ cm and $B: (10 \pm 1)$ cm

The best estimate for the sum of these two measurements is $20 \text{ cm} + 10 \text{ cm} = 30 \text{ cm}$

The smallest possible sum in adding these two measurements is $18 \text{ cm} + 9 \text{ cm} = 27 \text{ cm}$.

The largest possible sum in adding these two measurements is $22 \text{ cm} + 11 \text{ cm} = 33 \text{ cm}$.

Therefore, the range of possible sum ranges from 27 cm to 33 cm.

The sum can be expressed as (30 ± 3) cm.

Instead of going through the tedious steps like the above, apply the short-cut rule!

Verify that subtracting measurements works the same way.

When multiplying measurement by a numerical constant, multiply the uncertainty by the same constant.

$$Z = kA, \quad \text{the associated (absolute) uncertainty } \Delta Z = k\Delta A$$

- Think of the above as a special case of rule 1. For instance, $Z = 3A = A + A + A$
- For the case of dividing measurement by a numerical constant k , think of it as multiplying the inverse of the constant, $1/k$.
- Note that the relative uncertainty of Z is equal to that of A , as the constants cancelled in the ratios.

Example

The thickness of 100 pages of a book is measured to be 9.0 mm using an instrument with precision to 0.1 mm. Determine the thickness of a page with its associated uncertainty.

Thickness of 100 pages, $T = (9.0 \pm 0.1)$ mm

The average thickness of one page, $t = T/100 = 9.0 / 100 = 0.090000$ mm (leave in many d.p. first)

$$\Delta t = \frac{1}{100} \Delta T = \frac{1}{100} (0.1) = 0.001 \text{ mm} \quad (\text{to 1 s.f.})$$

Thus, $t \pm \Delta t = (0.090 \pm 0.001)$ mm = $(9.0 \pm 0.1) \times 10^{-2}$ mm (t expressed to same d.p as Δt)

Worked Example 1

Given the quantities: $x = 4.0 \pm 0.5$, $z = 5.36 \pm 0.04$

Determine the value of y in the following relationship: $y = x + 3z$

Solution:

$$\begin{aligned} y &= 4.0 + 3(5.36) \\ &= 20.08000000 \quad (\text{leave in many d.p. first}) \end{aligned}$$

$$\begin{aligned} \Delta y &= \Delta x + 3\Delta z \\ &= 0.5 + 3(0.04) \\ &= 0.6 \quad (\text{to 1 s.f.}) \end{aligned}$$

$$y \pm \Delta y = 20.1 \pm 0.6 \quad (y \text{ expressed to same d.p as } \Delta y)$$

2. Multiplication and Division of Measurements

When multiplying or dividing measurements, add their fractional uncertainties.

$$Z = A \times B \quad \text{or} \quad Z = A \div B$$

The relative uncertainty, $\frac{\Delta Z}{Z} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$ for both cases.

Suppose you need to multiply A: (20 ± 2) cm by B: (10 ± 1) cm.

The best estimate of the product is $20 \text{ cm} \times 10 \text{ cm} = 2.0 \times 10^2 \text{ cm}^2$

The smallest possible value would be $18 \text{ cm} \times 9 \text{ cm} = 1.6 \times 10^2 \text{ cm}^2$,

The largest reasonable value would be $22 \text{ cm} \times 11 \text{ cm} = 2.4 \times 10^2 \text{ cm}^2$.

The product must be $(2.0 \pm 0.4) \times 10^2 \text{ cm}^2$.

Check this rule for division for yourself.

Example:

A particle was found to travel (2.0 ± 0.2) cm in (3.0 ± 0.6) s. Determine the speed of the particle and its associated uncertainty.

$$v = s/t = 2.0/3.0 = 0.666667 \text{ cms}^{-1} \quad (\text{leave in many d.p. first})$$

$$\frac{\Delta v}{v} = \frac{\Delta s}{s} + \frac{\Delta t}{t} = \frac{0.2}{2.0} + \frac{0.6}{3.0} = 0.3$$

$$\text{So } \Delta v = 0.3 (0.6667 \text{ cms}^{-1}) = 0.2 \text{ cms}^{-1} \quad (\text{to 1 s.f.})$$

$$v \pm \Delta v = (0.7 \pm 0.2) \text{ cms}^{-1} \quad (v \text{ expressed to same d.p as } \Delta v)$$

The fractional uncertainty in A^n is n times the fractional uncertainty in A .

$$Z = A^n, \quad \text{The relative uncertainty, } \frac{\Delta Z}{Z} = |n| \frac{\Delta A}{A}$$

- Think of the above as a special case of rule 2. For instance, $Z = A^3 = A \times A \times A$, thus $\frac{\Delta Z}{Z} = 3 \frac{\Delta A}{A}$
- Note that for the case of negative powers, e.g. $Z = A^{-3}$, it is the same as $\left(\frac{1}{A}\right)^3$, thus $\frac{\Delta Z}{Z} = 3 \frac{\Delta A}{A}$, which is still positive.
- For the case of taking roots, e.g. $Z = \sqrt[n]{A}$, it can be written in the form: $A^{\frac{1}{n}}$, thus $\frac{\Delta Z}{Z} = \left|\frac{1}{n}\right| \frac{\Delta A}{A}$

Summary

1. If $Z = mA + nB$, then $\Delta R = |m|\Delta A + |n|\Delta B$ Note: m, n are numerical constants

2. If $Z = kA^m \times B^n$, where k is a numerical constant, then $\frac{\Delta Z}{Z} = |m| \frac{\Delta A}{A} + |n| \frac{\Delta B}{B}$

Example

Given that a sphere of radius $r = (18.5 \pm 0.5)$ mm, find the volume of the sphere with its associated uncertainties.

$$V = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi (18.5)^3 = 26522 \text{ mm}^3 \quad (\text{leave in many d.p. first})$$

$$\text{The fractional uncertainty of } V, \quad \frac{\Delta V}{V} = 3 \frac{\Delta r}{r} = 3 \left(\frac{0.5}{18.5} \right) = 0.081$$

$$\text{Absolute error of } V, \quad \Delta V = \frac{\Delta V}{V} \times V = 0.081 \times 26522 = 2000 \text{ mm}^3 \quad (\text{to 1 s.f.})$$

$$\text{Thus } V \pm \Delta V = (27000 \pm 2000) \text{ mm}^3 = (2.7 \pm 0.2) \times 10^4 \text{ mm}^3 \quad (V \text{ expressed to same d.p. as } \Delta V)$$

Worked Example 2

The formula for the period of a simple pendulum is $T = 2\pi \sqrt{\frac{l}{g}}$. Such a pendulum is used to

determine g . The fractional error in the measurement of the period T is $\pm x$ and in the measurement of the length l is $\pm y$. Assume there is no other sources error, what is the greatest fractional error in the calculated value of g ?

$$\text{Solution: } g = \frac{4\pi^2 l}{T^2} \quad \frac{\Delta g}{g} = \frac{\Delta l}{l} + 2 \frac{\Delta T}{T} = y + 2x$$

Mixtures of multiplication, division, addition, subtraction, and powers, and other functions

For complicated functions or special functions, Z

$$\begin{aligned} \text{Absolute uncertainty} &= \frac{1}{2} (\text{maximum possible value of } Z - \text{minimum possible value of } Z) \\ \Delta Z &= \frac{1}{2} (Z_{\max} - Z_{\min}) \end{aligned}$$

Or simply,

$$\begin{aligned} \text{Absolute uncertainty} &= \text{maximum possible value of } Z - Z \\ \Delta Z &= Z_{\max} - Z \end{aligned}$$

- The difference in methods may result in slightly different uncertainty results. All answers are acceptable as long as the method is logical.

Example

Consider $S = x \cos \theta$ for $x = (2.0 \pm 0.2)$ cm, $\theta = (53 \pm 2)^\circ$. Find S with its uncertainty.

$$S = (2.0 \text{ cm}) \cos 53^\circ = 1.204 \text{ cm}$$

To get the largest possible value of S
we would make x larger, $(x + \Delta x) = 2.2$ cm, and θ smaller, $(\theta - \Delta\theta) = 51^\circ$.

$$S_{\max} = (S + \Delta S) = (2.2 \text{ cm}) \cos 51^\circ = 1.385 \text{ cm.}$$

$$\Delta S = S_{\max} - S = 1.385 - 1.204 = 0.2 \text{ cm (to 1 s.f.)}$$

Then $S = (1.2 \pm 0.2)$ cm.

3 Scalars and vectors

AO (i) distinguish scalar and vector quantities and give examples of each.

- A scalar quantity is a physical quantity that can be expressed by a magnitude only. It DOES NOT have a direction. It is completely specified by its numerical value and unit.
- Scalar quantities can be added or subtracted using rules of algebra.

e.g. $10 \text{ kg sugar} + 12 \text{ kg sugar} = 22 \text{ kg sugar}$

- A vector quantity is a physical quantity that has BOTH magnitude and direction. It must be specified with its value, unit and direction.

e.g.: 2.5 ms^{-1} due East

↑
magnitude

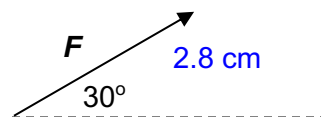
↙
direction

- In print, a vector is often denoted by a letter in bold type, e.g. force \mathbf{F} . In written form, it is either \vec{F} or \underline{f} . The magnitude is indicated as $|\mathbf{F}|$.
- A vector can be represented with an arrow whose length is proportional to the magnitude of the vector, is correctly orientated with respect to a reference direction



Scale: 2 cm represents 1 ms^{-1} .

The velocity is 2.45 ms^{-1} due East.



Scale 1 cm: 5 N

The force F is 14 N, in a direction 30° to the horizontal.

The table below shows a list of scalar and vector quantities:

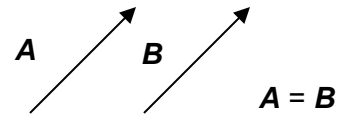
Scalar quantity
distance
speed
temperature
energy
power
mass
density
pressure
volume
time

Vector quantity
displacement
velocity
acceleration
force
momentum
weight
moment
torque
electric field
magnetic flux density

AO (j) add and subtract coplanar vectors

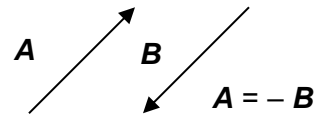
Equal vectors

Vectors are equal if they have the *same magnitude and direction*.



Negative vector

The negative of a vector has the *same magnitude but opposite direction*.



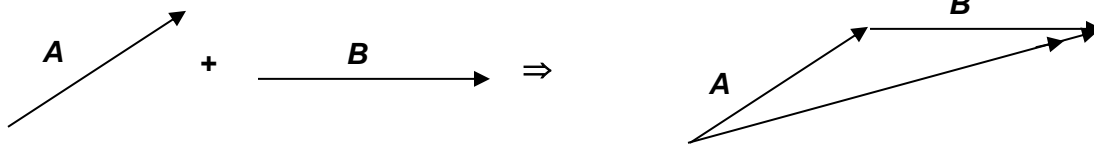
Coplanar vectors are vectors that lie on the same plane (2-dimension).

Unlike scalar quantities, vector quantities cannot be added or subtracted using algebra. Instead a *vector diagram* is needed.

(1) Vector Addition

a. Triangle method

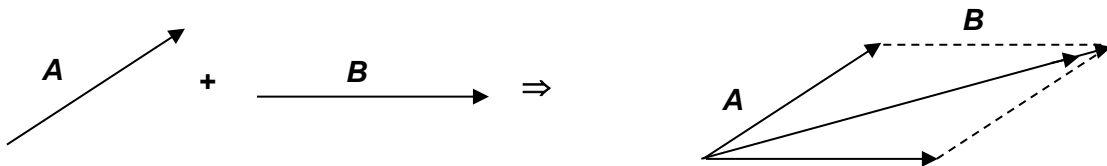
Stack the vectors such that one vector has its tail placed at the tip of the previous vector.



The resultant vector is represented by the line with a double arrow directed from the tail of the first vector to the tip of the last vector.

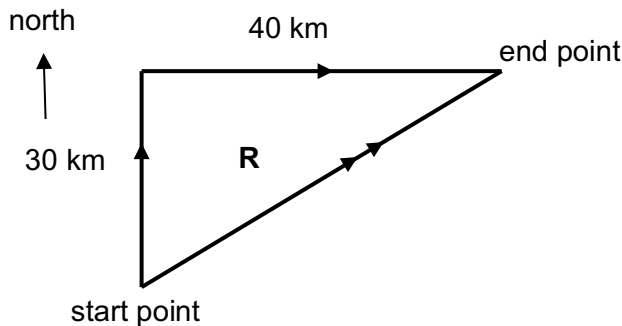
b. Parallelogram method.

Put the vectors to be added 'tail to tail'. Complete the parallelogram. The resultant vector is the diagonal from the tail of the two vectors to the other vertex of the parallelogram.

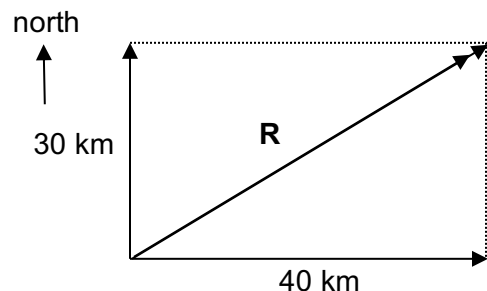


Example

Evaluate the resultant displacement of a ship which travels 30 km due north then 40 km due east.

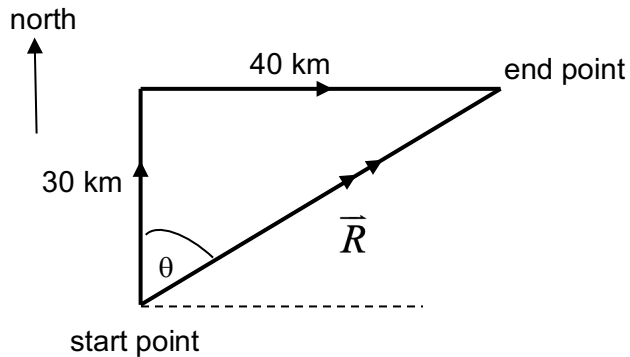


OR



Methods to determine the resultant magnitude and direction

- 1) Draw the vector to scale, measure the resultant magnitude using a ruler, and angle (for direction) using protractor.
- 2) (a) Use Pythagoras Theorem, trigonometric function to solve for right angle triangle.
(b) Use Sine Rule, Cosine Rule for irregular triangle.



Worked Solution:

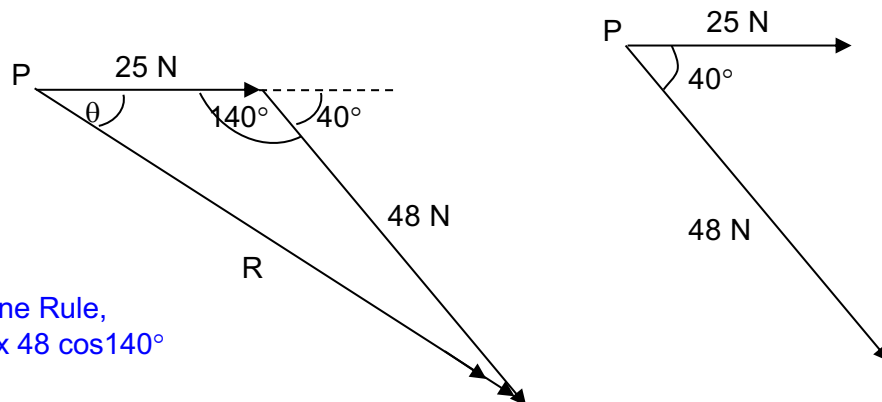
$$R = |\vec{R}| = \sqrt{40^2 + 30^2} = 50 \text{ km}$$

$$\theta = \tan^{-1}(40/30) = 53.1^\circ$$

The resultant displacement is 50 km, 53.1° East of North.

Practice:

Two forces act at a point P as shown below. Determine (magnitude and direction of) the resultant of these two forces.

Solution

Find magnitude by Cosine Rule,
 $R^2 = 25^2 + 48^2 - 2 \times 25 \times 48 \cos 140^\circ$
 $R = 69 \text{ N}$

Find the angle by Sine Rule

$$\frac{\sin \theta}{48} = \frac{\sin 140^\circ}{69}$$

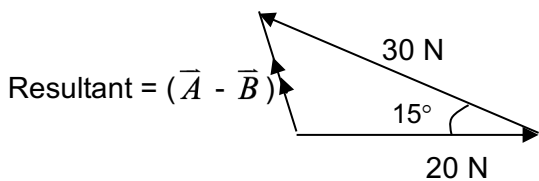
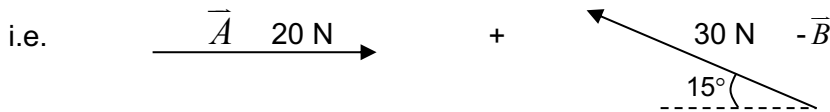
$$\theta = 26.6^\circ$$

Thus the resultant force is 69 N, 26.6° clockwise from the 25 N force.

(2) Vector subtraction

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

Example



Check

using cosine rule that $|\vec{A} - \vec{B}| = 12$ N,
using sine rule that it is 139° to the horizontal.

Vector subtraction can come about when we want to determine the change in a certain physical quantity.

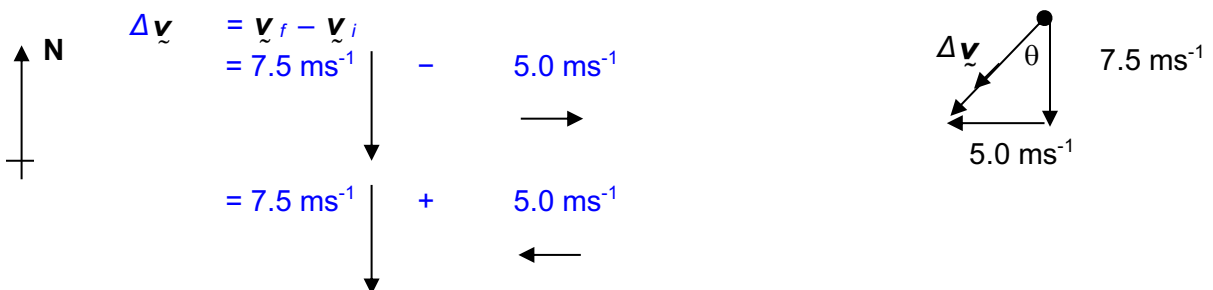
A change in a physical quantity $\Delta \vec{Q} = \text{final value } \vec{Q}_f - \text{initial value } \vec{Q}_i$

Mathematically it is written as $\Delta \vec{Q} = \vec{Q}_f - \vec{Q}_i$

Practice:

An object is moving at 5.0 ms^{-1} due east. Its direction changes to due south with a speed of 7.5 ms^{-1} . Determine the change in the velocity.

- (i) Change in velocity $\Delta \vec{v} = \vec{v}_f - \vec{v}_i$. A vector diagram is needed for the vector subtraction.



$$|\Delta \vec{v}| = \sqrt{5.0^2 + 7.5^2} = 9.0 \text{ ms}^{-1}$$

$$\theta = \tan^{-1} \frac{5.0}{7.5} = 33.7^\circ \quad \text{Thus, the change in velocity is } 9.0 \text{ ms}^{-1} \text{ with bearing } 213.7^\circ$$

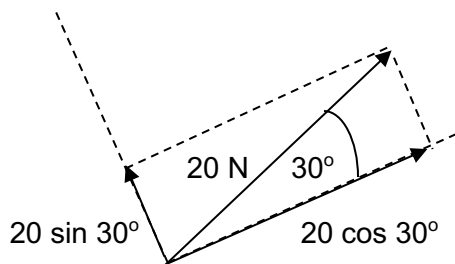
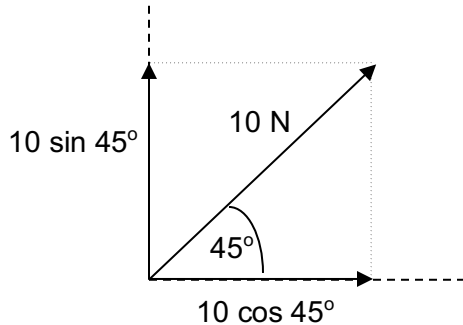
AO(k) represent a vector as two perpendicular components

Since two vectors can be added to give a resultant vector, any vector can be broken up (or resolved) into two vectors or components. It is more convenient to resolve a vector into two mutually-perpendicular components through the use of trigonometry and Pythagoras theorem. Mutually-perpendicular vectors are independent of each other.

Practice

For each given vector shown, draw the two perpendicular components in the direction given by the dotted lines and state their magnitudes in terms of the respective hypotenuses and angles.

example



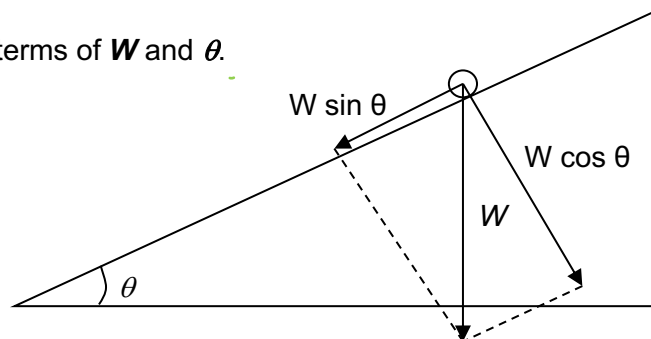
Each of the vectors above is resolved into two perpendicular components. A vector can be resolved into infinite pairs of perpendicular components. The choice of directions depends on the problem at hand.

Example 1:

An object rests on the plane of an inclined slope as shown. The weight W acts vertically down. Draw components of the weight

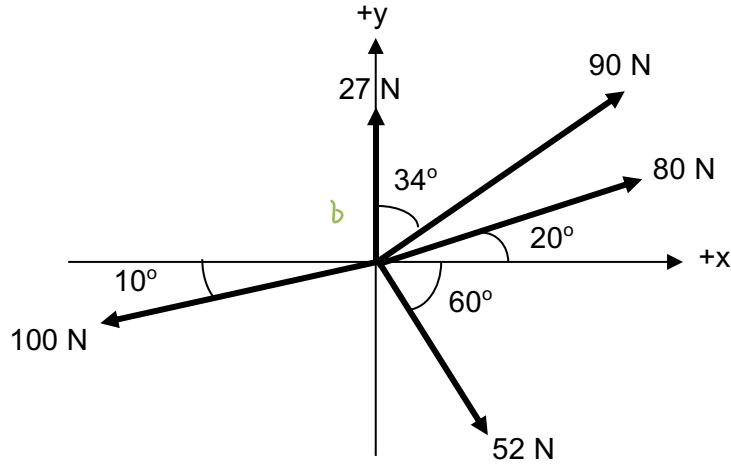
- parallel to the slope
- perpendicular (normal) to the slope.

Label the magnitude of the two components in terms of W and θ .



Example 2

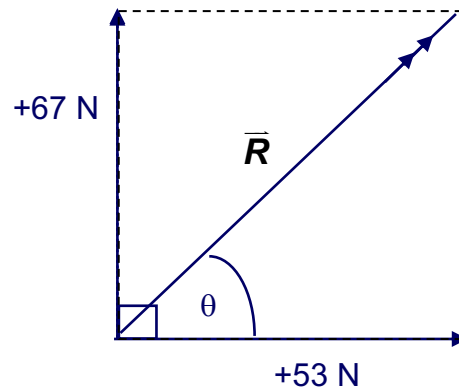
The 5 forces shown act on an object. Find the resultant force due to them.



- Resolve the vectors into two mutually perpendicular components:

Vector/N	x-component /N (+ →)	y-component /N (+↑)
100	$-100 \cos 10^\circ$	$-100 \sin 10^\circ$
27	0	27
90	$90 \sin 34^\circ$	$90 \cos 34^\circ$
80	$80 \cos 20^\circ$	$80 \sin 20^\circ$
52	$52 \cos 60^\circ$	$-52 \sin 60^\circ$
Resultant	53	67

- Draw vector diagram to show the resultant vector



- Compute the magnitude of the resultant vector

$$R = \sqrt{67^2 + 53^2} = 85 \text{ N}$$

- Indicate the direction with an angle with reference to a certain direction on the diagram and calculate the angle

$$\theta = \tan^{-1}(67/53)$$
$$= 52^\circ$$

- Write a complete statement to specify the magnitude and direction of the resultant vector

The resultant force is 85 N, 52° anticlockwise from +ve x axis.

Measurement Summary (Physical Quantities and Units)

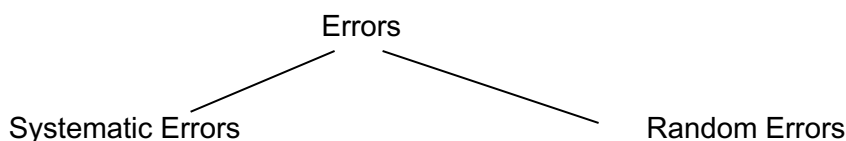
- | | | | | |
|----------|---|-----------------|------|------------------|
| Quantity | = | Numerical Value | Unit | e.g. mass = 5 kg |
|----------|---|-----------------|------|------------------|

Basics Quantity	=	Numerical Value	Basic Unit	e.g. time = 3 s
-----------------	---	-----------------	------------	-----------------

Derived Quantity	=	Numerical Value	Derived Unit	e.g. speed = 10 ms ⁻¹
------------------	---	-----------------	--------------	----------------------------------
- Basic Units: kg m s A mol K cd
- Steps to find unit of an unknown quantity
 - state the defining equation
 - make the unknown quantity the subject
 - derive the units of the unknown quantity in term of the basic units of the known quantities
 - simplify the units
- A physically correct equation must be homogeneous. However, not all homogeneous equations are physically correct.

Measurement Summary (Errors and Uncertainty)

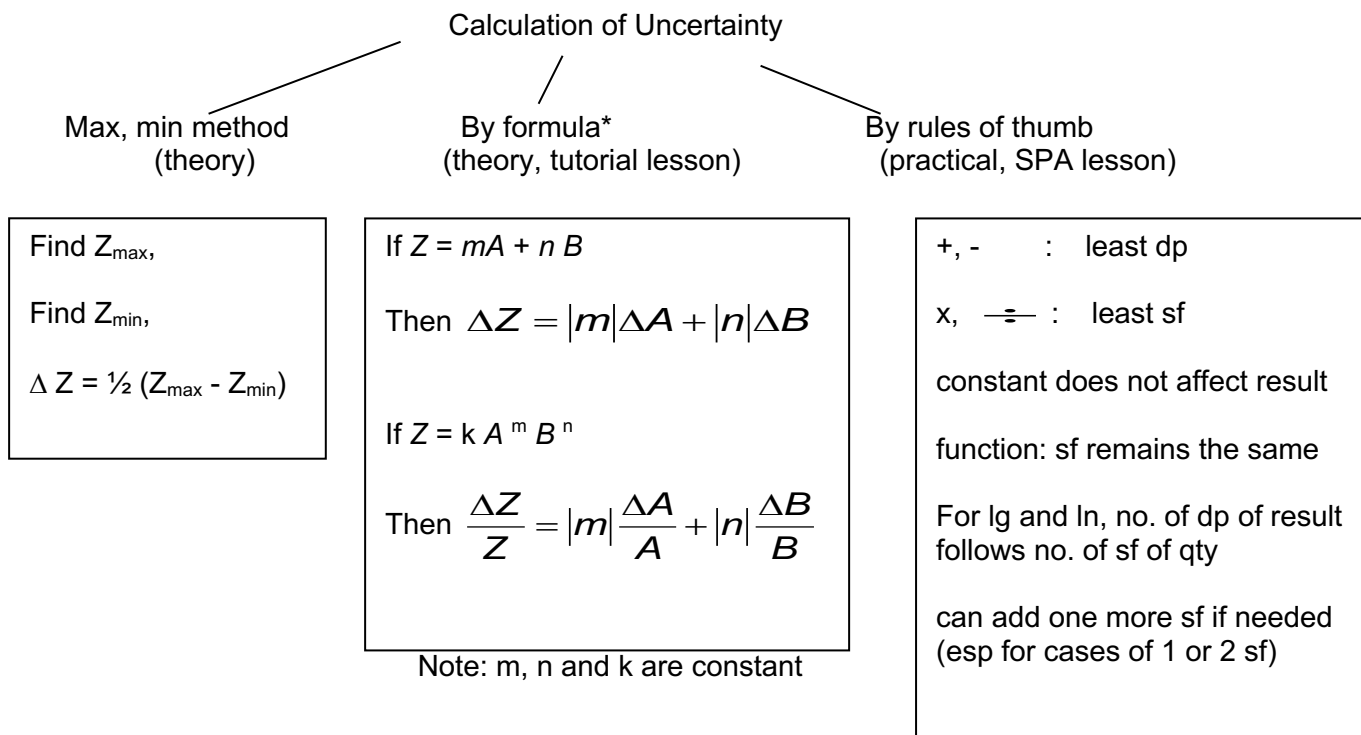
- due to factors like human, instrumental, techniques, environmental



If systematic error occurs, readings are not accurate (deviation between mean value and 'true' value is large)

If random error occurs, readings are not precise (scatter, range of readings is large)

Error leads to uncertainty



Steps to calculate the value of a physical quantity z with its uncertainty Δz

- (1) Express the unknown quantity as the subject.
- (2) Calculate the numerical value of the unknown quantity z without the uncertainty first (leave in many dp first).
- (3) Apply formula* to find the absolute uncertainty Δz of unknown quantity.
- (4) Express uncertainty Δz to 1 sf.
- (5) Round the quantity z to the same dp as the uncertainty Δz .
- (6) Express answer to standard form (not always compulsory but good practice)
- (7) Write the quantity with its uncertainty and the unit ($z \pm \Delta z$) unit
(Note: unit should be written outside bracket)

Fractional uncertainty: $\frac{\Delta Z}{Z}$

(usually expressed to 2 s.f.)

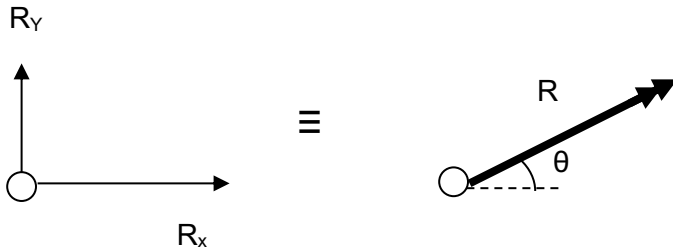
Percentage uncertainty: $\frac{\Delta Z}{Z} \times 100\%$

(usually expressed to 2 s.f.)

Measurement Summary (Vectors)

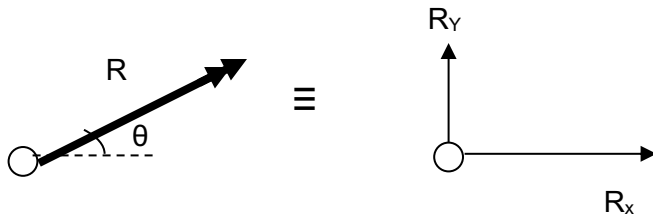
Scalar quantity has magnitude but no direction.
 Vector quantity has both magnitude and direction.

To find resultant



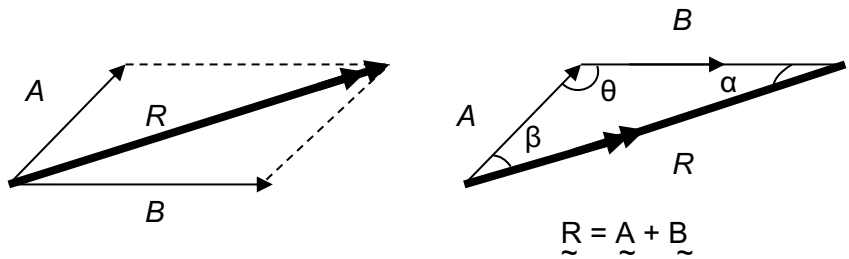
Note: $R^2 = R_x^2 + R_y^2$
 $\theta = \tan^{-1} (R_y / R_x)$

To resolve into components



Note: $R_x = R \cos \theta$
 $R_y = R \sin \theta$

Addition of vector (used for finding resultant)



$R^2 = A^2 + B^2 - 2AB \cos \theta$

$\frac{\sin \theta}{R} = \frac{\sin \alpha}{A} = \frac{\sin \beta}{B}$

To find change in velocity:

Step 1: $\Delta \underline{v} = \underline{v}_f - \underline{v}_i$
 $= \underline{v}_f + (-\underline{v}_i)$

Note: $-\underline{v}_i$ has same magnitude as \underline{v}_i but in opposite direction

Step 2: Apply vector addition to find $\Delta \underline{v}$ in term of both magnitude and direction.

Topic	Term/key word	Definition
Physical Quantities and Units Measurement Techniques	Physical Quantity	Any scientifically measurable quantities are called physical quantities. It has a numerical magnitude and may have a unit.
	Base quantities	The seven physical quantities of the SI system by which all other physical quantities are defined. They are: mass, length, time, temperature, amount of substance, electric current and luminous intensity.
	Base Units	The seven units of the SI system, related to the base quantities, whose magnitude is defined without referring to other units. They are: kg, m, s, K, mol, A and cd. Note: cd (unit for luminous intensity is not in the syllabus).
	Derived quantities	Are physical quantities that are derived from base quantities and can be expressed in terms of products and quotients of base quantities.
	Derived units	Are units that are derived from base units and can be expressed in terms of products and quotients of base units.
	Dimensionless quantity	One that is represented by a number and a unit not derived from base units
	Unitless quantity	It is a dimensionless quantity. It is represented only by a number
	Homogeneous Equation	Equation where units of all the terms are the same. Each term in the equation must correspond to the same physical quantity.
	Experimental errors	Uncertainties in measured quantities that arise from difference sources due to: (a) Limitations of observer (b) Measuring instrument used (c) Method used
	Random errors (2006)	Random errors are present when the measured readings are scattered about the <u>mean value</u> with <u>no fixed pattern</u> . They have equal probability of having different magnitudes and signs.
	Systematic errors (2006)	Systematic errors are consistent deviation of readings from the <u>true value</u> with a <u>fixed pattern</u> . They have the same magnitude and sign.

Precision #	The precision of a measurement is <u>how close</u> the experimental values are <u>to each other</u> . Precision is also a term used to describe the level of uncertainty in an instrument's scale.
Accuracy #	The <u>closeness</u> of experimental value <u>to the true value</u> of the quantity being measured.
Uncertainty	Uncertainty is the range of values on both sides of a measurement in which the actual value of the measurement is expected to lie. It is determined by the graduation of scale of the measuring instrument.
Absolute uncertainty	Absolute uncertainty depends on the separation between scale markings. Readings are usually taken to the smallest or half of the smallest scale graduation depending on the nature of the experiment and the instrument used.
Fractional uncertainty	Fractional uncertainty is the ratio of absolute uncertainty to the measured value of a quantity.
Percentage Uncertainty	Fractional uncertainty expressed in percentage form.
Vector (2009)	Vector is a physical quantity that has both <u>magnitude</u> and <u>direction</u> . It must be specified with its value, unit and direction.
Scalar (2009)	Scalar is a physical quantity that has only <u>magnitude</u> but does <u>not</u> have a <u>direction</u> . It is completely specified by its numerical value and unit.

APPENDIX A

SUMMARY OF KEY QUANTITIES, SYMBOLS AND UNITS

The following list illustrates the symbols and units that will be used in question papers.

Quantity	Usual symbols	Usual units	Quantity	Usual symbols	Usual units
Base Quantities					
mass	m	kg	electric current	I	A
length	l	m	thermodynamic temperature	T	K
Time	t	s	amount of substance	n	mol
Other Quantities					
distance	d	m	elementary charge	e	C
displacement	s, x	m	electric potential	V	V
area	A	m ²	electric potential difference	V	V
volume	V, v	m ³	electromotive force	E	V
density	ρ	kg m ⁻³	resistance	R	Ω
speed	u, v, w, c	m s ⁻¹	resistivity	ρ	Ω m
velocity	u, v, w, c	m s ⁻¹	electric field strength	E	N C ⁻¹ , V m ⁻¹
acceleration	a	m s ⁻²	permittivity of free space	ϵ_0	F m ⁻¹
acceleration of free fall	g	m s ⁻²	magnetic flux	Φ	Wb
force	F	N	magnetic flux density	B	T
weight	W	N	permeability of free space	μ_0	H m ⁻¹
momentum	p	N s	force constant	k	N m ⁻¹
work	w, W	J	Celsius temperature	θ	°C
energy	E, U, W	J	specific heat capacity	c	J K ⁻¹ kg ⁻¹
potential energy	E_p	J	molar gas constant	R	J K ⁻¹ mol ⁻¹
kinetic energy	E_k	J	Boltzmann constant	k	J K ⁻¹
heating	Q	J	Avogadro constant	N_A	mol ⁻¹
change of internal energy	ΔU	J	number	N, n, m	
power	P	W	number density (number per unit volume)	n	m ⁻³
pressure	p	Pa	Planck constant	h	J s
torque	T	N m	work function energy	ϕ	J
gravitational constant	G	N kg ⁻² m ²	activity of radioactive source	A	Bq
gravitational field strength	g	N kg ⁻¹	decay constant	λ	s ⁻¹
gravitational potential	ϕ	J kg ⁻¹	half-life	$t_{1/2}$	s
angle	θ	°, rad	relative atomic mass	A_r	
angular displacement	θ	°, rad	relative molecular mass	M_r	
angular speed	ω	rad s ⁻¹	atomic mass	m_a	kg, u
angular velocity	ω	rad s ⁻¹	electron mass	m_e	kg, u
period	T	s	neutron mass	m_n	kg, u
frequency	f	Hz	proton mass	m_p	kg, u
angular frequency	ω	rad s ⁻¹	molar mass	M	kg
wavelength	λ	m	proton number	Z	
speed of electromagnetic waves	c	m s ⁻¹	nucleon number	A	
electric charge	Q	C	neutron number	N	

APPENDIX B**Derivation of the rule: $\Delta Z = \Delta A + \Delta B$** **A For the case of $Z = A + B$,**

$$Z_{\max} = A_{\max} + B_{\max}$$

$$\begin{aligned} Z + \Delta Z &= (A + \Delta A) + (B + \Delta B) \\ &= (A + B) + (\Delta A + \Delta B) \end{aligned}$$

$$\text{Therefore } \Delta Z = \Delta A + \Delta B$$

B For the case of $Z = A - B$,

$$Z_{\max} = A_{\max} - B_{\min}$$

$$\begin{aligned} Z + \Delta Z &= (A + \Delta A) - (B - \Delta B) \\ &= (A - B) + (\Delta A + \Delta B) \end{aligned}$$

$$\text{Therefore } \Delta Z = \Delta A + \Delta B$$

APPENDIX C**Derivation of the rule: $\frac{\Delta Z}{Z} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$** **A For the case of $Z = A \times B$,**

Take the largest values for A and B, that is

$$\begin{aligned} Z + \Delta Z &= (A + \Delta A)(B + \Delta B) \\ &= AB + A \Delta B + B \Delta A + \Delta A \Delta B \end{aligned}$$

As $\Delta A \ll A$ and $\Delta B \ll B$,
the last term is much smaller than the other
terms and can be neglected.

$$\text{Thus } Z + \Delta Z = AB + (A \Delta B + B \Delta A)$$

$$\Delta Z = A \Delta B + B \Delta A$$

$$\text{the fractional uncertainty is } \frac{\Delta Z}{Z} = \frac{\Delta A}{A} + \frac{\Delta B}{B}.$$

B For the case of $Z = A \div B$,

$$\text{Maximum possible value of } Z + \Delta Z = \frac{A + \Delta A}{B - \Delta B}$$

$$\begin{aligned} \Delta Z &= \frac{A + \Delta A}{B - \Delta B} - \frac{A}{B} \\ &= \frac{BA + B\Delta A - AB + A\Delta B}{B(B - \Delta B)} = \frac{B\Delta A + A\Delta B}{B^2 - B\Delta B} \end{aligned}$$

$$\text{Since } B \Delta B \ll B^2, \Delta Z \approx \frac{B\Delta A + A\Delta B}{B^2}$$

Thus fractional uncertainty

$$\begin{aligned} \frac{\Delta Z}{Z} &= \frac{(B\Delta A + A\Delta B)}{B^2} \div \frac{A}{B} = \frac{(B\Delta A + A\Delta B)}{B^2} \times \frac{B}{A} \\ &= \frac{\Delta A}{A} + \frac{\Delta B}{B} \end{aligned}$$